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T3 ________________	Problem Chosen	F3 ________________
T4 ________________	A	F4 ________________

2016
MCM/ICM
Summary Sheet

(Your team's summary should be included as the first page of your electronic submission.)
Type a summary of your results on this page. Do not include the name of your school, advisor, or team members on this page.

We developed a model of the temperature of a hot water bath to assess and create a strategy so that one could enjoy a hot water bath that consumed the least amount of water to keep the water as close as possible to the initial temperature. To begin, we used Newton's Law of Cooling to examine the effects of the person's body temperature on the water as well as the effect of the ambient room temperature on the water temperature. Our model showed us that the main agent for loss of water temperature was caused by the ambient room temperature. Simulations illuminated that the most optimal method of keeping the water temperature steady was to add a hot amount of water to the bathtub if the temperature fell beneath a certain interval. This contradicts the idea of adding a continuous trickle of hot water to the system. Unfortunately, our model did not consider that the water added in increments would first have to propagate through heat diffusion. To this end, we used cellular automata to model the temperature of the water in small areas around the bathtub as water was added. But even with this model, it is still difficult to keep the water temperature as high and even as possible. From our initial analysis, in which the ambient room temperature was the main factor for the rapid decrease in water temperature, we propose the creative solution of increasing the temperature of the room to a level more comparable to the initial temperature of the bathtub and taking a shorter bath. Not only will this conserve the amount of water necessary for keeping the bathwater hot, it will also ease the rate at which the temperature of the water decreases.

Bathing Water Shalt Remain Hot!
A Computational Approach

Team #50890

Abstract

In this paper, we confront the issue of excessive usage of bathing water from a computational point. We designed a model based on the Newton's Law of Heating and Cooling, the laws of thermodynamics, and the concept of heat transfers between physical systems. To explain our model, we used iPython [1] computational tools to illustrate the results, weaknesses and strengths of our model. Our initial approach assumed that the heat propagated throughout the water instantaneously. To combat this, we used a cellular automaton to model the heat distribution throughout the bathtub.

[1] iPython is an interactive computing tool that offers advanced and improved ways of running mathematical simulations on a computer.

Contents

1 Introduction

Taking a hot bath can be relaxing and rejuvenating, but of course, all of us want to keep the bathwater as hot as possible while minimizing the use of additional hot water. To this end, we have developed a model in which we optimize a hot bath so that one can enjoy the warmth of the water while conserving the amount of additional hot water added. In the following sections, we propose our reasoning as to what effects the temperature of the water the most, and we examine several ways to combat the falling temperature of the water.

2 Assumptions for our Model

For the sake of simplicity, we assumed the following:

• The bathtub in our model has no thermal conductivity, thus the only possible heat transfers happening between distinct physical systems are:
(1) Between the human body and the bathing water.
(2) Between the bathtub environment (everything in the bathtub) and the surrounding air.

• The hot water's faucet and the overflow drain are positioned on opposite positions of the bathtub. That way, we know for sure that the incoming heat from the faucet is being diffused throughout the whole bathtub before exiting on the other end's overflowing drain.

To keep our results consistent, we used the following constant values throughout our computations:
(1) The average mass of a human body is 62 kg, and the temperature of a normal healthy person is $37°C$.
(2) Average room temperature: $21°C$. According to BBC, $21°C$ is the recommended room temperature[2].
(3) Average bath temperature: $38°C$[3], one degree above the human body temperature as according to Rachel Forder of the Telegraph Magazine, bath temperatures should be slightly warmer than the average human temperature.

[2]"http://news.bbc.co.uk/2/hi/health/5372296.stm"
[3]Forder, Rachel. "So Mr Prescott, how hot should my bath be?"

3 Mathematical Calculations and Simulations

3.1 Newton's Law of Cooling

3.1.1 A Person's Effect on Water Temperature

Our first approach is to examine the effects of the person in the bathtub to the temperature of the water as a closed system (assuming no loss of heat to the surroundings). We model the heat transfer rate based on on Newton's Law of heating and cooling which states that the rate of change of the temperature with respect to time of an object is proportional to the difference between the temperature of the object and the temperature of its surroundings. So from Newton's Law, we get the following equation:

$$mc\frac{dT}{dt} = -hA(T - T_H) \tag{1}$$

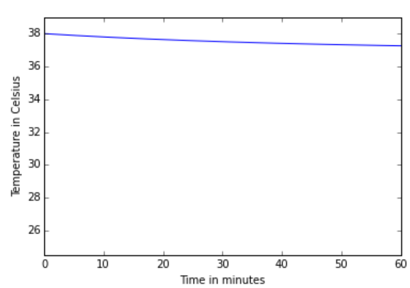
where m, h, A, and T_H are the human mass, skin heat transfer coefficient, surface area, and body temperature respectively. c is the specific heat capacity of water. Redistributing this equation yields

$$\frac{dT}{(T - T_H)} = -\frac{hA}{mc}dt \tag{2}$$

This is an exponential decay function with the solution

$$T(t) = (T_o - T_H)e^{-\frac{hA}{mc}t} + T_H \tag{3}$$

We calculated and plotted values of our solution using the matplotlib [4] Python



library. It is evident that even after an hour in the bathtub, there is minimal

[4] A 2D plotting library that is included in the Python programming language.

heat transfer between the body and the water assuming that this is a closed system.

As it is shown in **Figure 1**, the very presence of a person inside the bathtub filled with hot water does not cause the water to cool down that much assuming that it is a closed system. Hence, to determine the effect of room temperature on hot water, we can consider the bathub water and the person inside it to be one entity.

3.1.2 Room Temperature's Effect on Hot Water

We will now explore a system in which the bathtub is exposed to ambient room temperature. Once again, we utilize Newton's Law of Cooling

$$mc\frac{dT}{dt} = -hA(T - T_a) \tag{4}$$

In this case,

$$T(t) = (T_o - T_a)e^{-\frac{hA}{mc}t} + T_a \tag{5}$$

Each constant is now attributed to the water in the bathtub, and T_a is the ambient room temperature which is assumed to be 21^oC. The following plot illustrates the exponential decay in temperature caused by the large difference in the temperature of the water and the room temperature.

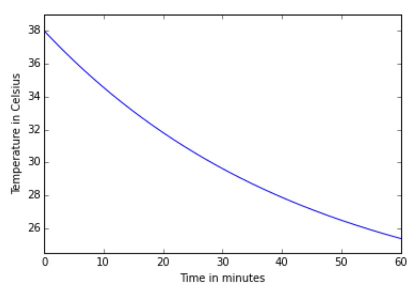


Figure 1: Now, we can safely conclude that the room temperature is the main cause for temperature loss in the bathtub.

3.2 Keeping Bathing Water Evenly Hot: Model's Computational Goal

To keep the water currently in the bathtub hot, hotter water must be added from the faucet. But as the water is being added, to avoid overfilling - excess water will flows out via an existing overflow drain. In this case, as mentioned in our model's assumptions we will have three distinct systems:

• System 1: Hot water coming from the faucet with mass m_1 and temperature t_1.

• System 2: Bathing water that is already in the bath with mass m_2, and temperature t_2.

• System 3: Excess bathing water that is exiting through the overflow drain, with mass m_1, and temperature t_2.

Simulation Assumptions:

1. We will consider the excess bathing water that is exiting through the overflow drain to have the same temperature as the remaining bathing water at any given point of time.
2. We will also assume that the exiting water has exactly the same mass as the incoming hot water at any given time.

3.3 Heat Energy Transfer: The Balance of Masses

Using the energy transfer formula, we calculated the equilibrium temperature at any given time, t, using the following formula:

$$m_1c_1(t_1 - t_f) = m_2c_2(t_f - t_2) \tag{6}$$

As the $c_1 = c_2$ (both are the values of the special heat capacity of water), we simplify the above equation to get this:

$$m_1(t_1 - t_f) = m_2(t_f - t_2) \tag{7}$$

By solving the above equation for the value of t_f, we got:

$$t_f = \frac{m_1 t_1 + m_2 t_2}{m_1 + m_2} \tag{8}$$

After using 50^o degrees Celsius as our set temperature for the hot water coming from the faucet, and setting the bathing water temperature to be equal to the equilibrium temperature of the bathing environment, and assuming that 82 kg (82 L is the common capacity of regular bathtub, we computed subsequent values of t_f over time; then we plotted the resulting values of bathing water's temperature versus time that has passed after t_0.

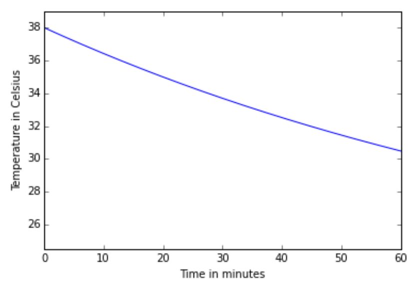


Figure 2: As it can be seen, a constant trickle of hot water can keep the water's temperature warm, but it doesn't keep water temperature within within a comfortable range.

4 Improving our Model - Minimizing Water Usage

After realizing that it is almost impossible to keep the bathing water warm enough without wasting extra hot water in addition to the amount that was already in the bathtub. Knowing that, anyone who owns a bathtub is presented with three choices:

• Reduce time spent in bathtubs.
• Keep the temperature in the bathroom relatively close to that of the water in the bathtub.
• Find an efficient ways to add hot water to the bathtub while keeping the temperature relatively high compared to our previous attempts.

From the above, we chose **finding efficient ways to use less water while trying to maintain high temperatures** to be the only choice that is feasible, from both a financial and environmental-friendly point of view.

After running several simulations, we found out that to keep water warm enough by using a lesser amount of water, we must not add hot water as a constant trickle, but in relatively large quantities every-time the bathtub environment temperature falls beyond a given temperature. But of course this is not realistic since the person taking the bath would need to have some kind of alarm that tells them when to let in more hot water from the faucet.

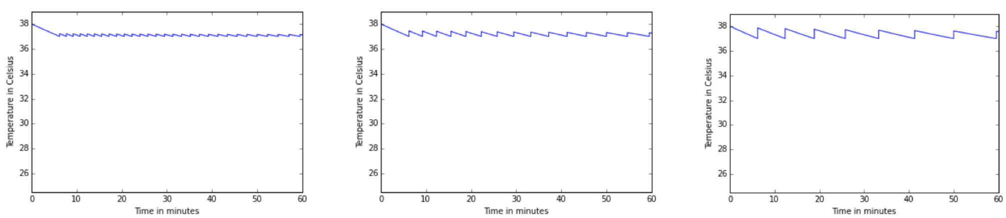


Figure 3: Amount of water in Liters added to a bathtub with an initial temperature of 38^o C containing 82 liters of water. From left to right: (1) 74.3 added in 32 rounds times, (2) 76.8 liters added in 16 rounds, and (3) 82 liters added in 8 rounds. during any bathing period.For technical reasons, we set one bathing period to last for 60 minutes.

After looking at the results above, we found out that temperature over a certain time is positively related to the number of rounds hot water has been added to the bathtub; thus even if the total amount of extra hot water added to the bathtub is decreasing, **our model predicts that increments of water are more effective in keeping the water warm enough than adding hot water as a one continuous constant trickle.**

Now that we know that increments are more efficient at keeping bathing water warm enough, we can improve our model by using calculations to determine the amount of water added to the bathtub based on the current bath temperature, while trying to maximize it at the same time.

To do that we had to establish a relation between the d_t, desirable temperature, c_t, the current temperature, m_w, the mass of water water that is already in the bathtub, and δ, a time dependent constant.

$$m_a = m_w * (c_t/d_t) * \delta \tag{9}$$

Using the above formula in which we determine the mass to be added m_a at any given time during our simulation, we got new simulation results, which we plotted in the graph below.

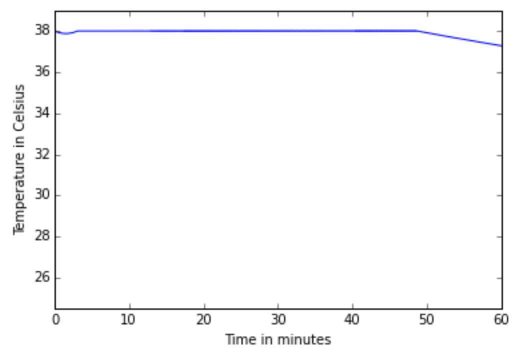


Figure 4: So far this is most efficient strategy as it manages to keep the temperature close to the desirable target while using a relatively an acceptable amount of water when compared to our previous simulations.

Now that we established the most efficient way to add hot water to the bathtub system, we can add a little bit more water to the system so that we can maximize the bathing water temperature for the whole duration of the session.

By tweaking on the parameters a little bit, we found out that by just adding a small surplus of hot water we can maintain our desirable temperature the whole time. For example, let's say we start with a human body laying in a bathtub. Let's say the weight of the human body is equal to 62 kg (average weight of an adult male), and that, without overflowing, the bathtub is capable of holding 82 liters of water plus the human body.

In the above figure, the desirable temperature is 38 degrees Celsius(one degree above that of the human body); maintaining that temperature would require adding almost 85 liters of hot water at 50 degrees Celsius in one hour

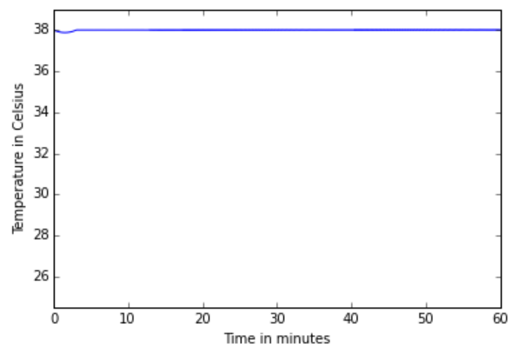


Figure 5: The results of the simulation based on the above example.

time period, and that is if there is an automatic system that detects the temperature in the bathtub and and releases an appropriate amount of hot water from the faucet in accordance to formula (9).

5 Sensitivity Analysis

When it came to testing our model, we established there are three major factors that would definitely affect our model in a considerable way.

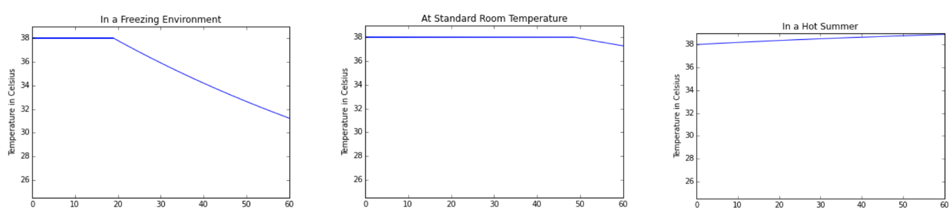


Figure 6: In this figure, one notices that in a colder environment, it is more difficult to keep the bathtub's temperature from falling down; and that in hot weather, if the surrounding air is hotter than the bathing water, the bathing water average temperature will actually increase, even the need of adding extra hot water from the faucet.

5.1 Temperature of the Surroundings

First, the average temperature of the bathroom, or the air's ambient temperature if the bathtub was located outside, would significantly affect our results as the bathtub system is in direct contact with the surrounding environment. For example, this is how the last optimized results would look like in three different thermal conditions.

5.2 Duration of a Normal Bath

In our simulations, we assumed that a bath takes an hour, but what if we changed that? Would that have effects on our model results?

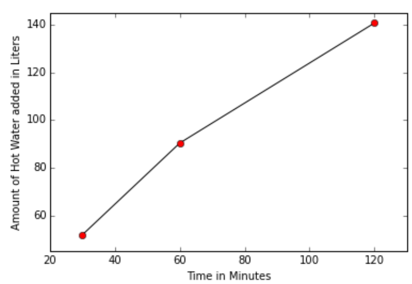


Figure 7: This plot illustrates the impact that the duration of the bath has on the amount of water used, that is why it should be preferable for one to take short baths, if he or she plans on avoiding wasting water.

5.3 Bathtub Size Variations

Lastly, we considered what changes different designs of bathtubs would have on our results. As mentioned in our assumptions, we assumed that bathtub was incapable of transferring heat, and that once all water molecules in the bathtub had the same temperature independent of their position. Thus according to our model, the shape and the presumed position of the faucet and the overflow drain have no effect whatsoever on our results. However, the size of the bathtub was accounted for in our computations, thus we analyzed what could there would be any notable changes to our model anticipated results.

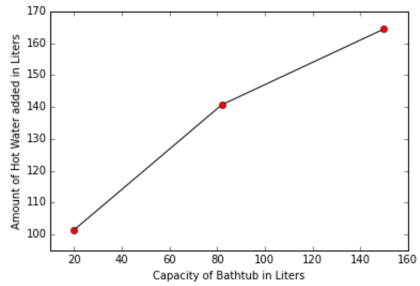


Figure 8: Our conclusion from the computer simulations as represented in this figure is that larger bathtubs are more effective in keeping the water hot for longer, but still the total amount of water used adds up to be the same as even though larger baths requires small proportional amounts of added hot water, they require large amounts of pre-heated water to begin with at t_0.

6 Assessment of our Model

6.1 Strengths and Weaknesses of our Model

In this section, we recognize the strengths and weaknesses of our model.

6.1.1 Strengths

1. With many methods of modeling, we utilized Occam's Razor to use the simplest model to explain the decrease in temperature of the water.

2. Our model is based on Newton's Law of Heating and Cooling which gets straight to the heart of the main issue: the temperature loss of water is due to the surroundings.

6.1.2 Weaknesses

1. Our model assumes that once hot water from the faucet is added to the water already in the bathtub, the water inside the bathtub instantly becomes slightly warmer as well which is not actually the case. Recognizing this limitation, we used a different approach that takes heat diffusion of water into account in the next section.

6.2 A Slightly Better Approach: Cellular Automaton Diffusion Simulations

As mentioned above in section **6.1.2**, our model doesn't take into account the heat diffusion of added hot water from the faucet. So to get around that, we used cellular automaton to model the heat diffusion of water inside the bathtub. To simplify the situation, we assume that each internal point on a line perpendicular to the top surface has the same temperature as the temperature on the top surface. For the cellular automaton itself, each cell represents the temperature of bathtub water at that point, and at each time iteration we calculate the new temperature of the cell based on Newton's Law of Heating and Cooling. Hence, if a cell has temperature T at time t, then the same cell will have temperature $T + \triangle T$ at time $t + \triangle t$ where the change in temperature $\triangle T$ is equal to the diffusion rate parameter (r) times the sum of each difference in the temperature of a neighbor $(MooreNeighbor_i)$ and the cell's temperature as well as the difference of the cell's temperature and the room temperature. The following equation shows how we calculate the difference in temperature for each time iteration:

$$\triangle T = r \sum_{i=1}^{8} (MooreNeighbor_i) + (T_s - T) \tag{10}$$

where $0 < r < 1/8 = 0.125$ and T_s is the room temperature.

For the initial conditions, we keep the same assumptions that a normal hot bath is about $38°C$ and the ambient room temperature $T_s = 21°C$. For the boundary conditions of our cellular automaton, we assumed the absorbing boundary conditions where all cells at the end rows and columns have neighbors with constant temperature which makes sense since the bathtub itself has a constant temperature.

So we did our simulations by constantly applying hot water at one column end which is equivalent to assuming that the faucet is located at one end of the bathtub and we also varied the volume of the bathtub by changing the size of our cellular automaton for multiple simulations.

6.2.1 Adding a Bubble Bath Additive

Using the cellular automaton simulations, we understood the effect of the bubble additive to be a decrease in the diffusion rate at which heat propagates throughout the bathtub. So by decreasing the diffusion rate r in equation (10) and running our simulations over and over, we determined that the chemicals inside the bubble additive slows the heat diffusion of water a little bit down which would in turn require to add more hot water into the bathtub. So our best advice is to not use too much bubble additive if we want to conserve water.

7 Conclusion

While our model may have been simple with several assumptions, it was evident that the biggest contributor to temperature loss inside the bathtub came from the fact that ambient room temperature is much lower than the bath water temperature. It was also evident that the difficulty in maintaining the water temperature even throughout the bath stemmed from the loss of heat to the surrounding as well as the nature of heat diffusion of water inside the bathtub. Thus, our creative solution to conserving the most amount of water were the following:

- Before bathing, increase the room temperature.

- Take a shorter bath, and try to be closer to the faucet as that is where hot water will be coming from.

- If you add water, do not add at a constant trickle, but rather add the water in larger quantities over a certain time interval.

Given more time, our model can be made even more accurate by modeling the heat diffusion of water in 3D since the cellular automaton simulations assume the heat diffusion of water in 2D.

8 The Cost of a Hot Bath: A Letter!

Dear Bathtub Owner,

We modeled your bathtub to see the most efficient way to keep the temperature of the water to your likings while using the least amount of additional hot water.

Our model told us that you are not causing the temperature of the water to decrease by much, but that the temperature of the room is the main agent behind the cooling water. While it may seem appropriate to add a constant trickle of water to your bathtub to increase the temperature, our simulations have shown us that by adding a large amount of water at set intervals, your water will actually stay hotter! We also suggest that you stay as close to the source of the hot water as possible and make sure that the overflow drain is located far from the source. This is because the temperature of the water near the faucet will be the hottest, while the overflow drain will take out the coldest water.

But we do have another solution for you, if you are willing. If your goal is to conserve water and keep the temperature of the water as close as possible to the start of bath, then we suggest that you first increase the temperature of the room closer to the temperature of the water. Not only will this ensure that you enjoy a relaxing and calming bath, it will also keep the temperature of the water hotter for a longer period of time, so you donâĂŹt have to add as much water as you would otherwise. We hope that you take use of our suggestions and see an improvement in the quality of your hot baths as well as keeping a few bucks in your pocket.

Yours for the sake of a hot bath,

The Bathtub Research Team (#50890)

References

[1] Raymond A. Serway, John W. Jewett, Jr. *Physics for Scientists and Engineers*. Brooks Cole, 9th Edition, 2013.

[2] Michelle Roberts. *Why more people die in the winter*. BBC News,*http://news.bbc.co.uk/2/hi/health/5372296.stm*", 2006.

[3] Rachel Forder, *So Mr Prescott, how hot should my bath be?* `http://www.telegraph.co.uk/news/health/3318578/So-Mr-Prescott-how-hot-should-my-bath-be.html`, 2005.